Lecture 4 - Variants of Bandit Problems

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September 19, 2022

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Introduction

Clarification : About the base of log terms

- Base of log terms is mostly not important in this course.
- We are concerned with the leading terms i.e., whether the regret bound is in terms of T or $\sqrt{T}$ or $\log T$ and not with constants.

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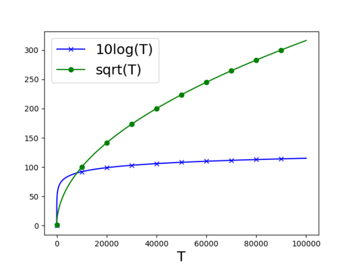

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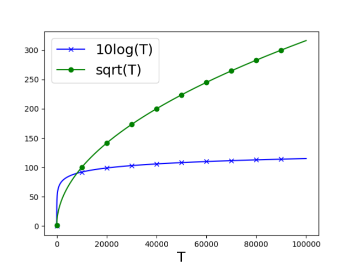

- You can convert the base of a *log* from e to 2 or 10 (and vice versa) just with an extra multiplicative constant (see here).
- So the base only affects the constant in the results, and hence it is mostly not important.

A Quick Recap of Lecture 1, 2 and 3

- Lecture 1 : Introduction to reinforcement learning and its basic elements.
- Lecture 2 : UCB for stationary stochastic bandits and its regret bound. Frequentist perspective.
- Lecture 3 : Thompson sampling for stationary stochastic bandits and its regret bound. Bayseian perspective.

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Stationary Stochastic Bandits

- Number of arms $= K$.
- Reward for arm $a \sim X_a$ with mean μ_a.
- $X_1, X_2, \ldots X_K$ are unknown stationary distributions.
- At each time step $t = 1, \ldots, T$, the agent,
  - chooses an arm $i(t)$, and
  - receives a numerical reward $r(t) \sim X_{i(t)}$.

Assumptions in our Bandits Model so far

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Lecture 4: Outline

- Non-stationary Stochastic Bandits.
- Adversarial Bandits.
- Dueling Bandits (and a Lower Bound)
- Contextual Bandits.

Non-stationary Stochastic Bandits

Non-stationary Reward Distributions

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- Rewards are assumed to be generated by a stochastic process.
- Learner can only select one arm at a time and sees absolute feedback.
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Non-stationary Stochastic Rewards

- Number of arms $= K$.
- At time step t, Reward for arm $a \sim X_a(t)$ with mean $\mu_a(t)$.
- For some t's, $\mu_a(t) \neq \mu_a(t+1)$.
- How could we characterize these changes?

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  - Number of change-points $\leq M$.

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  - Mean rewards change at unknown time steps called change-points and remain constant between two change-points.
  - Number of change-points $\leq M$.
- **Bound the variation in mean rewards.**
  - Mean rewards can change an arbitrary number of times, but total variation is bounded i.e.,

$$\max_a \sum_{t=1}^{T-1} |\mu_a(t) - \mu_a(t+1)| \leq V.$$

Algorithm for Non-stationary Stochastic Bandits (with a bound on the number of changes)

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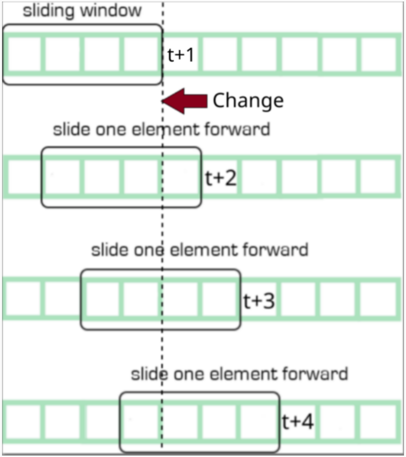
- $\tau =$ size of the window.

Algorithm Sliding Window-UCB algorithm [Garivier and Moulines, 2011]

for $t = 1, \ldots, K$ **do**
    Choose each arm once.
end for
for $t = K + 1, \ldots$ **do**
    Compute empirical means $\hat{\mu}_1(t-1), \ldots, \hat{\mu}_K(t-1)$ based on last τ time steps.
    Select arm $i(t) = \arg\max_a \left[\hat{\mu}_a(t-1) + \text{confidence term}\right]$.
end for

How Does Sliding Window-UCB Work?



- After a change occurs, sliding window forgets the past and considers history from the current setting.

Adversarial Bandits

Adversarial Bandits

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- ~~Rewards are generated by a stochastic process~~ Adversarial bandits.
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Adversarial Rewards: Simple Example

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3. The adversary observes the selected arm and secretly chooses rewards,
    - for arm 1, reward x_1 from $\{0, 1\}$, and
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No matter what the learner does, the adversary can always cause linear regret for the learner.

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What happens if the the learner's policy is deterministic?

Oblivious Adversarial Rewards: Deterministic Policy

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- If the learner implements a deterministic policy e.g., play arm 1,

  the adversary can choose $x_1 = 0$ and $x_2 = 1$, since the adversary knows the learner's policy and,

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- Deterministic policies cause linear regret!

Oblivious Adversarial Rewards: Randomized Policy

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- If the learner implements a randomized policy (e.g., play arm 1 with probability 0.5),

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  the learner's expected regret $= \max\{x_1, x_2\} - \mathbb{E}[r] = 1/2$.

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- Randomized policies can achieve sub-linear regret.☺

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- Performance measure?

Performance Measure : Regret I

- Recall that for stationary stochastic bandits, the goal was to minimize

$$\mathfrak{R}_\pi(T) := \underbrace{T\mu^*}_{\text{Optimal expected cumulative reward}} - \underbrace{\mathbb{E}\left[\sum_{t=1}^{T} r(t) \mid \pi\right]}_{\text{Expected cumulative reward of } \pi} .$$

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- Competing with the policy that always plays the best arm?

Performance Measure : Regret II

- The benchmark policy : 'always play the best arm (in hindsight)',

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- The cumulative reward of the benchmark policy $= \max_a \sum_{t=1}^{T} x_a(t)$.
- The learner's goal is to minimize the expected cumulative regret.

$$\mathfrak{R}_\pi(T) := \max_a \sum_{t=1}^{T} x_a(t) - \mathop{\mathbb{E}}_{\pi}\left[\sum_{t=1}^{T} r(t) \mid \pi\right]$$

Algorithm for Adversarial Bandits : EXP3

- Assigns weight to each arm.
- Higher weight $\implies$ higher selection probability.
- γ = exploration parameter.
- Weight of the selected arm is updated via an estimator.
- Arms producing more rewards receive higher weights.

Algorithm EXP3 [Auer et al., 2003]

For each arm a, initialize $w_a(1) = 1$.

At time t, for each arm a,

$$p_a \leftarrow (1-\gamma)\frac{w_a(t)}{\sum_{b=1}^{K} w_b(t)} + \frac{\gamma}{K}$$

Pick $i(t) \sim \mathbf{p(t)} = (p_1, \ldots, p_K)$.
Receive reward $r(t) := x_{i(t)}(t)$.
for $a = 1, 2, \ldots, K$ **do**

$$\hat{x}_a(t) = \begin{cases} \frac{r(t)}{p_a(t)} & \text{if } a = i(t) \\ 0 & \text{otherwise} \end{cases}$$

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5: **for** $a = 1, 2, \ldots, K$ **do**

6: $\hat{x}_a(t) = \begin{cases} \frac{r(t)}{p_a(t)} & \text{if } a = i(t) \\ 0 & \text{otherwise} \end{cases}$

7: $w_a(t+1) \leftarrow w_a(t) \exp\left(\frac{\gamma}{K}\hat{x}_a(t)\right)$.

8: **end for**

Algorithm for Adversarial Bandits: EXP3

- Assigns weight to each arm.
- Higher weight $\implies$ higher selection probability.
- $\gamma =$ exploration parameter.
- Weight of the selected arm is updated via an estimator.
- Arms producing more rewards receive higher weights.

Algorithm EXP3 [Auer et al., 2003]

For each arm a, initialize $w_a(1) = 1$.

At time t, for each arm a,

$$p_a \leftarrow (1-\gamma)\frac{w_a(t)}{\sum_{b=1}^{K} w_b(t)} + \frac{\gamma}{K}$$

Pick $i(t) \sim \mathbf{p(t)} = (p_1, \dots, p_K)$.

Receive reward $r(t) \coloneqq x_{i(t)}(t)$.

for $a = 1, 2, \dots, K$ **do**

$$\hat{x}_a(t) = \begin{cases} \frac{r(t)}{p_a(t)} & \text{if } a = i(t) \\ 0 & \text{otherwise} \end{cases}$$

$$w_a(t+1) \leftarrow w_a(t) \exp\left(\tfrac{\gamma}{K}\hat{x}_a(t)\right).$$

end for

Regret Bound for EXP3

Theorem (Auer et al. [2003])

The expected cumulative regret of EXP3 is $\mathcal{O}\left(\sqrt{TK\log(K)}\right)$.

Key Lemma in the Regret Analysis

Let "history"

$$\mathcal{F}_t := i(1), i(2), \ldots, i(t-1).$$

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$\mathbb{E}[\hat{x}_a(t) \mid \mathcal{F}_t] = x_a(t).$

- $\hat{x}_a(t)$ estimates the reward of arm a at time t.

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Proof.

$$\mathbb{E}[\hat{x}_a(t)]$$
$$= \left[p_a(t) \cdot \frac{x_a(t)}{p_a(t)} + (1 - p_a(t)) \cdot 0\right]$$
$$= x_a(t)$$

□

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Break

We start again after a break.

Dueling Bandits

Dueling Bandits

- Reward distributions are stationary.
- Rewards are assumed to be generated by a stochastic process.
- ~~Learner can only select one arm at a time and sees absolute feedback~~ Dueling bandits.
- Learner has no extra information about the arms.

Feedback to the Learner?

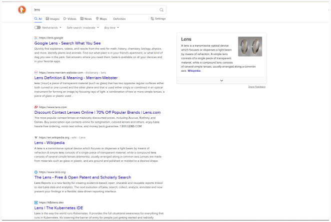
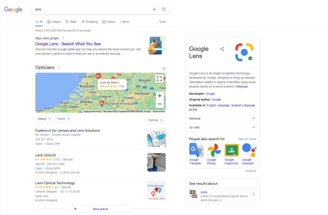


Figure 1: DuckDuckGo search results



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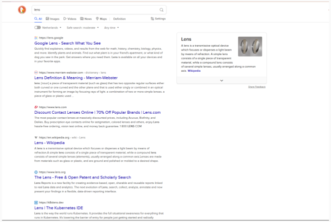

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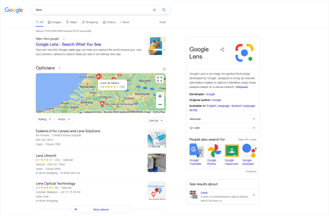

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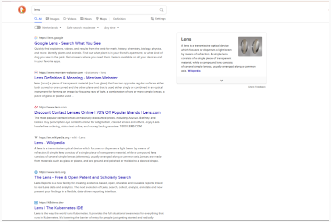

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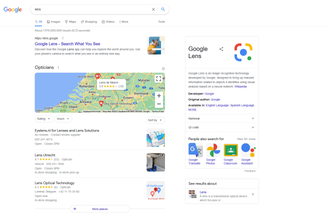

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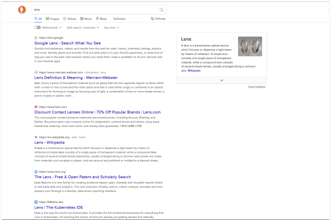

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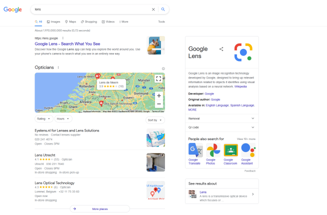

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- Relative feedback by interleaved filtering [Radlinski and Joachims, 2007]

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(where $i(t)$ and $j(t)$ are the arms picked at time t).

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- Lower bound for stationary stochastic bandits $= \sqrt{KT}$ [Auer et al., 2003]

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- Lower bound for dueling bandits = Lower bound for stationary stochastic bandits.

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Algorithm Reduction from stationary stochastic bandits

```
Repeat
1: (i, j) ← DBA.decide(t).
2: x_i(t) ← CBE.get_reward(i).
3: x_j(t + 1) ← CBE.get_reward(j).
4: DBA.update(t, (i, j), ψ(x_i − x_j)).
5: t = t + 2.
Until t ≥ T
```

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- $\mathbb{E}$[Regret of DBA] is of the same order as $\mathbb{E}$[Regret of CB].

Algorithm for Adversarial Dueling Bandits: REX3

- Assigns weight to each arm.
- Higher weight $\implies$ higher selection probability.
- $\gamma \in (0, 0.5)$ exploration parameter.
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- Arms winning more duels receive higher weights.

Algorithm REX3 (PG et al.)

1: For each arm a, initialize weights

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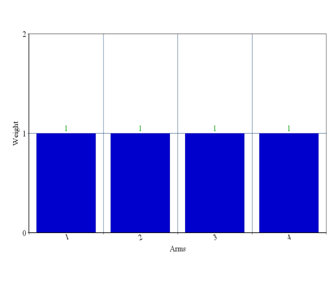
How Does REX3 Work?

Weights at $t = 0$
($\gamma = 0.4$)



- Update weight according to (relative) feedback.

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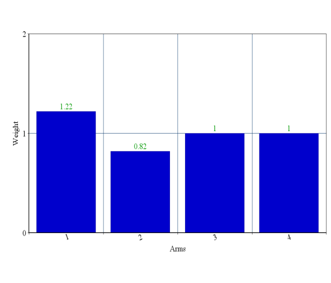
How Does REX3 Work?

$i = 1$, $j = 2$

1 wins the duel

Weights at $t = 1$



- Weight may decrease.

Algorithm REXЗ

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How Does REX3 Work?

$i = 1, j = 3$

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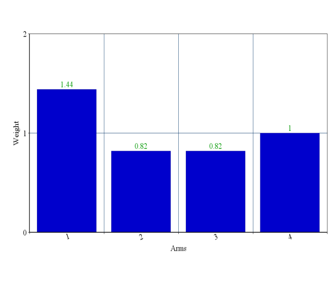

- Weights increase at arms which win regularly.

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Theorem (PG et al., 2015)

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- The upper bound is only $\sqrt{\log(K)}$ away than the lower bound $\sqrt{KT}$ so $\mathrm{REX3}$ is near-optimal.

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- When when arms i and j are selected,

$$\text{feedback} = f = \begin{cases} -1 & \text{if } x_i < x_j \\ 0 & \text{if } x_i = x_j \\ +1 & \text{if } x_i > x_j \end{cases}$$

Estimator for an arm

Algorithm REX3

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$\mathbb{E}[\hat{c}_a(t) \mid \mathcal{F}_t] =$
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- $\hat{c}_a(t)$ estimates the relative utility/advantage of picking arm a instead of picking an arm according to $\mathbf{p}(\mathbf{t})$.

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$$\begin{aligned}
\mathop{\mathbb{E}}_{(i,j)\sim\mathbf{p(t)}}[\hat{c}_a(t)] &= \sum_{m=1}^{K}\sum_{n=1}^{K} p_m p_n \left[\mathbb{I}[a = m]\frac{(x_m - x_n)}{2p_m} + \mathbb{I}[a = n]\frac{(x_n - x_m)}{2p_n}\right] \\
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&\quad + \sum_{m=1}^{K}\sum_{n=1}^{K} p_m \cancel{p_n} \ \mathbb{I}[a = n]\frac{(x_n - x_m)}{2\cancel{p_n}} \\
&= \sum_{n=1}^{K} p_n \frac{(x_a - x_n)}{2} + \sum_{m=1}^{K} p_m \frac{(x_a - x_m)}{2} \\
&= \mathop{\mathbb{E}}_{i\sim\mathbf{p(t)}}[x_a(t) - x_i(t)].
\end{aligned}$$

Contextual Bandits

Contextual Bandits

- Reward distributions are stationary.
- Rewards are assumed to be generated by a stochastic process.
- Learner can only select one arm at a time and sees absolute feedback.
- ~~Learner has no extra information about the arms.~~ Contextual bandits.

Availability of Extra Information

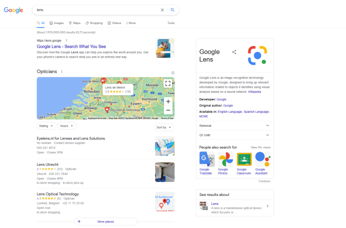

Figure 3: Google search results

- Observation of extra information (*context*) before choosing an action.
- Practical scenario : News recommendation, ad selection.

Contextual Bandits: Mathematical Setting

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  - the learner chooses an arm $i(t)$ and,
  - the learner receives a reward $r(t) = r_{t,i(t)}$.
- Linear dependence: $\mathbb{E}[r_{t,a} \mid \mathbf{x_t}] = \mathbf{x_t} \cdot \theta_a$ for some unknown vector $\theta_a \in \mathbb{R}^d$.

Example

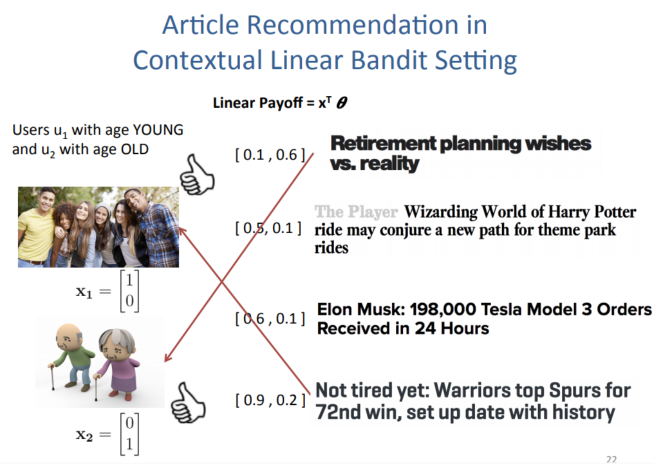

Image source: *blogpost*

Algorithm for Linear Contextual Bandits

Algorithm LinUCB [Li et al., 2010]

1: Compute confidence regions $C_{a,t}$ for each arm a.
2: Observe feature vector (context) $\mathbf{x_t} \in \mathbb{R}^d$.
3: For each arm a, compute

$$UCB_t(a \mid \mathbf{x}_{t,a}) = \sup_{\hat{\theta}_a \in C_{a,t}} \mathbf{x_t} \cdot \hat{\theta}_a$$

4: Select the arm which maximizes $UCB_t(a \mid \mathbf{x}_{t,a})$

Summary

- Non-stationary Stochastic Bandits.
- Adversarial Bandits.
- Dueling Bandits (and a Lower Bound)
- Contextual Bandits.

Next Lecture

- Reinforcement learning in Markov decision processes.
- A near-optimal algorithm : UCRL.

References

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